Algebraic Geometry Lecture 32 – Sheafification Continued

Joe Grant[1]

1. Sheafification

The following proposition illustrates the local nature of sheaves.

Proposition. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Then φ is an isomorphism if and only if the induced map*

$$\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$$

is an isomorphism for every $P \in X$.

Sketch proof. Use property (4) to get injectivity, and properties (4) and (5) to get surjectivity. Hartshorne has the details. □

Note that this proposition is false, in general, for presheaves. For example, it's easy to define a morphism $\varphi : \mathcal{F}_2 \to \mathcal{F}_3$ of presheaves which is an isomorphism on stalks but clearly isn't an isomorphism of presheaves.

Now we have some understanding of morphisms of sheaves we want to define kernels, images, etc. The obvious definitions are $U \mapsto \ker \varphi(U)$ and $U \mapsto \operatorname{im} \varphi(U)$ with induced restriction maps. One can show that the kernel of a morphism of sheaves, as defined above, is a sheaf, but with the above definition the image is only a presheaf (because "gluing doesn't commute with morphisms of sheaves"). This leads to the following definition.

Definition. For any presheaf $\mathcal{F}$ on X we define the sheafification $\mathcal{F}^+$ of $\mathcal{F}$ to be the following: For any open set $U \subseteq X$, let $\mathcal{F}^+(U)$ be the set of all functions $s : U \to \bigcup_{P \in U} \mathcal{F}_P$ such that, for all $P \in U$,

1. $s(P) \in \mathcal{F}_P$,
2. there is a neighbourhood $V \subseteq U$ of P and an element $t \in \mathcal{F}(V)$ such that for all $Q \in V$, the germ $t_Q = s(Q)$.

Proposition. *$\mathcal{F}^+$ with the natural restriction maps is a sheaf and there exists a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ such that for any sheaf $\mathcal{G}$ and any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \psi\theta$. Furthermore, the pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism.*

Example. $\mathcal{F}_2^+(X_i) = \{s : \{x_i\} \to \mathbb{Z}\}$, $\mathcal{F}_2^+(X) = \mathcal{F}_2^+(X_1) \oplus \mathcal{F}_2^+(X_2)$. $\mathcal{F}_2^+$ is clearly isomorphic to $\mathcal{F}_3$.

2. Sheaves of modules (and maybe some algebraic geometry)

Definition. A ringed space $(X, \mathcal{O}_X)$ is a topological space X equipped with a sheaf $\mathcal{O}_X$ of rings, which we call the structure sheaf. Think of $\mathcal{O}_X$ as the sheaf of "good functions" on X.

Let $V \subseteq k^n$ be an affine algebraic set. We want to define "good functions" on the open sets of V. We'll be guided by two things:

1. the good functions on V should be the regular functions $k[V]$.
2. V has a very simple basis of open sets, the sets $D(f)$.

With a little work one can show that it's enough to define the structure sheaf on a basis of open sets if it satisfies some natural gluing and restriction conditions.

[1]Typed by Lee Butler based on a talk by Joe to the G-izzle.

Definition. Let V be an affine algebraic set and consider a nonzero $f \in k[V]$. Then we set

$$\mathcal{O}_V(D(f)) = k[V]_f,$$

the localisation of $k[V]$ along $\{f^n : n \in \mathbb{N}\}$, with the obvious restriction maps. This defines a sheaf of rings on V called the sheaf of regular functions.

The stalks $\mathcal{O}_{V,P} = (\mathcal{O}_V)_P$ of regular functions are just the local rings of P on V, as defined in previous lectures.

Definition. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ on X such that for each open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and the restriction maps are $\mathcal{O}_X(U)$-module maps.

If V is open in U we have two restriction maps

$$r : \mathcal{O}_X(U) \to \mathcal{O}_X(V), \qquad \rho : \mathcal{F}(U) \to \mathcal{F}(V).$$

$\mathcal{F}(U)$ is an $\mathcal{O}_X$-module by definition and $\mathcal{F}(V)$ becomes one via the map r. Then requiring that restriction maps are $\mathcal{O}_X(U)$-module maps means that ρ must be $\mathcal{O}_X(U)$ equivariant, i.e. for $f \in \mathcal{O}_X(U)$ and $m \in \mathcal{F}(U)$,

$$\rho(fm) = r(f)\rho(m).$$

(This is for left modules.)

Example. If $(X, \mathcal{O}_X)$ is $(\{x_1, x_2\}, \mathcal{F}_3)$ then an $\mathcal{O}_X$-module $\mathcal{F}$ is given by two $\mathbb{Z}$-modules, i.e. abelian groups, $\mathcal{F}(X_1)$ and $\mathcal{F}(X_2)$. We must have $\mathcal{F}(X) = \mathcal{F}(X_1) \oplus \mathcal{F}(X_2)$ and the restriction maps of $\mathcal{F}$ are the obvious projections.

Given an affine algebraic set V there are some particularly nice $\mathcal{O}_V$-modules.

Definition. Let M be an $\mathcal{O}_V$-module. We define an $\mathcal{O}_V$-module $\widetilde{M}$ on the standard open sets of V by

$$\widetilde{M}(D(f)) = M_f := (\mathcal{O}_V(V))_f \otimes_{\mathcal{O}_V(V)} M.$$

In particular, $\widetilde{M}(V) = M$.

Definition. An $\mathcal{O}_V$-module isomorphic to $\widetilde{M}$ for some $\mathcal{O}_V$-module M is called quasi-coherent. If M is also finitely generated over $\mathcal{O}_V(V)$ we say $\widetilde{M}$ is coherent.

Note that $\mathcal{O}_V$ is itself always an $\mathcal{O}_V$-module. (So are $0, \mathcal{O}_V \oplus \mathcal{O}_V$, etc.) It is coherent as $\mathcal{O}_V = \widetilde{\mathcal{O}_V(V)}$.